Dimensional Reduction of Geometric Moduli in Warped String Compactifications

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I. Introduction & Motivation

- 10d superstring theory → (Low energy limit) → 10D supergravity + stringy ingredients

  ↓ Compactify

  4d particle physics

- Ideally:
  Start with the 10d superstring theory. Next, compactify. Next, take low energy limit.

- We will take the first route. Focus will be: type IIB superstring theory.

One slide for the system:

Low energy limit: we have 10d type IIB SUGRA (pseudo)action:

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2}\left[\int d^{10}X\sqrt{-g}\; R_g + \frac{1}{2}\int d\Phi\wedge\star d\Phi + \frac{1}{2}\int e^{-\Phi}H_3\wedge\star H_3 \right.$$
$$\left. + \frac{1}{2}\int e^{2\Phi}F_1\wedge\star F_1 + \frac{1}{2}\int e^{\Phi}\tilde{F}_3\wedge\star\tilde{F}_3 + \frac{1}{4}\int \tilde{F}_5\wedge\star\tilde{F}_5\right]$$
$$+ \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3\,,$$

+ action for local sources like D-branes, O-planes (objects with -ve tension).

g_{MN} : 10D metric in Einstein frame, B_2 : Kalb-Ramond 2-form, Φ : dilaton, C_0, C_2, C_4 : R-R forms.

$$H_3 = dB_2\,,\quad F_1 = dC_0\,,\quad F_3 = dC_2\,,\quad F_5 = dC_4\,,\quad \tilde{F}_3 = F_3 - C_0H_3\,,\quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3\,.$$

$\tau \equiv C_0 + ie^{-\Phi}$ [axiodilaton], $G_3 \equiv F_3 - \tau H_3$.

- 10d type IIB SUGRA, in presence of D-branes and O-planes, admits GKP backgrounds.

[Giddings-Kachru-Polchinski ’01]

- These backgrounds feature: a 10d spacetime = a warped 4d Minkowski spacetime × a 6D conformally CY orientifold.

Warp factor

$$ds^2_{(0)} = e^{2\Omega^{(0)}} e^{2A^{(0)}(y)} \hat{\eta}_{\mu\nu} dx^\mu dx^\nu + e^{-2A^{(0)}(y)} \tilde{g}^{(0)}_{mn}(y) dy^m dy^n$$

Ricci flat internal manifold

- 3-form fluxes $F_3^{(0)}, H_3^{(0)}$ source the warp factor: $\tilde{\nabla}^{\tilde{2}} e^{-4A^{(0)}} \sim |G_3^{(0)}|^2 \sim \alpha'^2$. [$G_3^{(0)}$ can be expressed in terms of 3-form fluxes which are quantized in units of α'.]

- In LVS, the effective action from a Minkowski × CY compactification is a good approximation; warp factor and flux corrections are $\mathcal{O}(\alpha'^2)$.

- A GKP background can be deformed by parameters u_A, yielding a family of 10d solutions. Upon compactification, they lead to massless fields in 4d.

$$\mathscr{L}_{\text{4d}} \propto G^{AB}(\vec{u})\partial_\mu u_A \partial^\mu u_B$$

- We aim to compute the moduli space metric $G^{AB}(\vec{u})$, even in case of strong warping.

- Our focus: **geometric moduli** which include all Kähler deformations along w/ certain complex structure deformations of the internal CY metric. These moduli have not been dimensionally reduced previously.

- Earlier dimensional reductions focused on: universal volume (Kähler) modulus, axions descending from C_4, C_2, B_2 form fields, and D3 brane position moduli. [Frey-Torroba-Underwood-Douglas ’08, Frey-Roberts ’13, Cownden-Frey-Marsh-Underwood ’16]

- In a different approach, the Kähler potential for geometric Kähler moduli has been derived from a combined consideration of expected $N = 1$ SUSY properties of the 4d effective theory and the structure of 10d vacua. [Martucci ’16]

- The dimensional reduction methods rely on the 10d equations of motion. They apply whether GKP background breaks SUSY [i.e. $G_3^{(0)}$ includes a $(0,3)$ component] or not. Can apply to more general compactifications.

Why warped (flux) compactifications?

- Similar as Randall-Sundrum scenario: $M_{\text{pl}}^2 = M_s^8 \times$ warped 6d volume $\quad [\Omega^{(0)} = 0$ case]

$$\frac{1}{\kappa_{10}^2} \sim \frac{1}{\alpha'^4} \sim \frac{1}{l_s^8} \sim M_s^8$$

- A classical mechanism for stabilizing some of the geometric complex structure moduli.

- Non-perturbative corrections could stabilize all other moduli $\Longrightarrow$ phenomenological applications in cosmology [string-pheno].

Outline for the remaining talk:

II. Geometric moduli space in GKP compactifications

III. Dimensional reduction, 4d moduli space metric

IV. Outlook

II. Geometric moduli space in GKP compactifications

$$ds^2_{(0)} = e^{2\Omega^{(0)}} e^{2A^{(0)}(y)} \hat{\eta}_{\mu\nu} dx^\mu dx^\nu + e^{-2A^{(0)}(y)} \tilde{g}^{(0)}_{mn}(y) dy^m dy^n \,,$$

$$\tilde{F}^{(0)}_5 = e^{4\Omega^{(0)}} \hat{\epsilon} \wedge \tilde{d} e^{4A^{(0)}(y)} + \tilde{\star}^{(0)} \tilde{d} e^{-4A^{(0)}(y)} \,,$$

$$\tilde{\star}^{(0)} G^{(0)}_3 = i G^{(0)}_3 \,, \quad \tau^{(0)} = \text{constant} \,,$$

$$-\tilde{\nabla}^{\tilde{2}} e^{-4A^{(0)}} = \frac{|G^{(0)}_3|^2}{2\text{Im}\tau^{(0)}} + \text{contributions from local sources} \,.$$

GKP background

- $\tilde{F}^{(0)}_5$ is self-dual [to match the low-energy limit of type IIB superstring theory and to ensure closure of 10D SUSY algebra].

- $G^{(0)}_3$ features components only on the compact directions + Harmonic + **Imaginary self-dual (ISD)** w.r.t. background CY orientifold.

- Consider linear metric fluctuations $u\delta\tilde{g}_{mn}(y)$ around the background CY metric. u: constant parameter

  1. ensuring Ricci flatness at linear order.
  2. preserving unwarped volume at linear order: $\tilde{g}^{mn}_{(0)}\delta\tilde{g}_{mn} = 0$ [tracelessness].

- On top of these, consider covariantly transverse fluctuations: $\tilde{\nabla}^{\tilde{m}}\delta\tilde{g}_{mn} = 0$ [will be required later for varying $u(x)$, otherwise a gauge choice].

- $\delta\tilde{g}_{mn}$ can be written in terms of specific harmonic forms: [Candelas-de la Ossa '91]

  — harmonic $(1,1)$ forms [Kähler deformations]
  — harmonic $(2,1)$ and $(1,2)$ forms [complex structure deformations]

  Moduli space of CY

- We further want to go from one GKP solution to another, by a metric deformation of above kind. This will allow few of the above deformations. Geometric moduli space of GKP compactification

- In response, allow fluctuations:

$$A(y) = A^{(0)}(y) + u\delta A(y)\,, \qquad \Omega = \Omega^{(0)} + u\delta\Omega\,, \qquad \tau = \tau^{(0)} + u\delta\tau\,,$$

$$\delta_u G_3 = u\chi_3(y)\,, \qquad \chi_3(y) \equiv \tilde{d}\eta_2 + \delta\tau \frac{G_3^{(0)} - \bar{G}_3^{(0)}}{2i\mathrm{Im}\tau^{(0)}}\,,$$

$$\delta_u \tilde{F}_{[0,5]} = -\,u\,\tilde{\star}^{(0)}\,V_1 + u\,\tilde{\star}^{(0)}\,\tilde{d}\delta e^{-4A} \quad \text{[self-dual at linear order]}\,.$$

- We want the deformed 10d system to remain a solution of GKP type. The perturbed G_3 must be ISD at linear order.

$$\Longrightarrow \boxed{W_3 = \tilde{d}\eta_2 + i\,\tilde{\star}^{(0)}\,\tilde{d}\eta_2 + \frac{i\delta\tau}{\mathrm{Im}\tau^{(0)}}\bar{G}_3^{(0)}} \quad \text{"linearised ISD"}$$

- Definitions used above: $V_m = \delta\tilde{g}_{mn}\partial^{\tilde{n}} e^{-4A^{(0)}}\,, \qquad W_{mnp} \equiv 3\delta\tilde{g}_{l[m}G^{(0)\tilde{l}}{}_{np]}\,.$

$$W_3 = \tilde{d}\eta_2 + i\,\tilde{\star}^{(0)}\,\tilde{d}\eta_2 + \frac{i\delta\tau}{\mathrm{Im}\tau^{(0)}}\bar{G}_3^{(0)} \qquad [\bar{G}_3^{(0)} : \text{harmonic } (1,2) + (3,0)]$$

- For **any** Kähler deformation we deduce $\delta\tau = 0$. The linearised ISD condition can be satisfied.

- Depending on the background flux $G_3^{(0)}$ and a complex structure deformation, the linearised ISD condition can be satisfied **only if** the harmonic piece in W_3 cancels the $\delta\tau$ piece on RHS. Refer to as **flat** complex structure deformation. Axiodilaton remains flat $[\delta\tau \neq 0]$.

- complex structure deformations - those violate linearised ISD - are stabilised. They are beyond the scope of current work

- $\delta A(y), \delta\Omega, \delta\tau, \eta_2(y)$ are found in order to solve linearised 10d SUGRA equations, in both cases above [Kähler and **flat** complex structure].

- Some non-geometric moduli of GKP compactification: b_I. $\delta C_4 \supset b_I\,\tilde{\star}^{(0)}\,\omega_2^I$, where ω_2^I forms basis of harmonic $(1,1)$ forms

III. Dimensional reduction, 4d moduli space metric

- 6D metric fluctuation: $u(x)\delta\tilde{g}_{mn}(y)$ allowing x-dependence. $u(x)$: 4d field. $\delta\tilde{g}_{mn}(y)$: either a Kähler deformation or a **flat** complex structure deformation

- First we need to find **correct** 10d fluctuations. Second substitute those into the 10d action, and integrate y

- Our ansatz:

$$ds^2 = e^{2\Omega+2A}\hat{\eta}_{\mu\nu}dx^\mu dx^\nu + 2e^{2\Omega+2A}\partial_\mu u B_m dx^\mu dy^m + e^{-2A}\tilde{g}_{mn}dy^m dy^n\,,$$

$$\tilde{F}_5 = e^{4\Omega}\hat{\epsilon}\wedge\tilde{d}e^{4A} + \tilde{\star}\tilde{d}e^{-4A} + e^{4\Omega}\,\hat{\star}\,\hat{d}u\wedge B_1\wedge\tilde{d}e^{4A} - e^{4\Omega+4A}\,\hat{\star}\,\hat{d}u\wedge\tilde{d}B_1 - e^{2\Omega}\hat{d}u\wedge\tilde{\star}\tilde{d}B_1\,,$$

$$G_3 = G_3^{(0)} + \hat{d}u\wedge\eta_2 + u\chi_3\,,\quad \tau = \tau^{(0)} + u\delta\tau\,,$$

$$\Omega = \Omega^{(0)} + u\delta\Omega\,,\quad A = A^{(0)} + u\delta A\,,\quad \tilde{g}_{mn} = \tilde{g}_{mn}^{(0)} + u\delta\tilde{g}_{mn}\,.$$

- Expanding in u and its derivatives, zeroth order terms coincide with GKP background values. At linear order, self duality of $\tilde{F}_5$ has been maintained.

- Satisfying 10d Einstein field equations, equations of motion for the form fields, and Bianchi Identities, at linear order we get equations of types:

$$1. \quad [\cdots]\, u = 0\,, \qquad 2. \quad [\cdots]\, \partial_\mu u = 0\,, \qquad 3. \quad [\cdots]\, \partial_\mu \partial_\nu u = 0 \quad \mu \neq \nu\,,$$

$$4. \quad [\cdots]\, \hat{\Box}\, u = 0\,, \quad \hat{\Box} \equiv \hat{\eta}^{\mu\nu}\partial_\mu \partial_\nu\,.$$

- Eq (4) will only be satisfied on-shell: $\hat{\Box}\, u = 0$

- We set terms proportional to u as zero. This has been done in the previous section. The deformed 10d system remain on moduli space of GKP compactification.

- Satisfying equations of first three types leads to constraints on: $\delta A(y), \delta\Omega, \eta_2(y), \delta\tau, B_m(y)$. We will impose them off-shell, as important for 10d gauge and diffeomorphism invariances.

Results holding in both geometric settings [Kähler and flat complex structure]:

$$\tilde{\nabla}^{\tilde{m}}\delta\tilde{g}_{mn}\,, \qquad \tilde{d}\,\tilde{\star}^{(0)}\,\eta_2 = 0, \qquad \delta\Omega = 0\,,$$

$$\tilde{\nabla}^{\tilde{2}}\delta e^{-4A} = \tilde{\nabla}^{\tilde{m}} f_m\,, \qquad \tilde{\nabla}^{\tilde{2}} B_m = e^{-2\Omega^{(0)}} f_m\,, \qquad [\delta e^{-4A} \equiv -4e^{-4A^{(0)}}\delta A]\,,$$

$$f_m = \delta\tilde{g}_{mn}\partial^{\tilde{n}} e^{-4A^{(0)}} - \frac{1}{4\mathrm{Im}\tau^{(0)}}\left[\eta_{np}\bar{G}_3^{(0)}{}_m{}^{\widetilde{np}} + \bar{\eta}_{np}G_3^{(0)}{}_m{}^{\widetilde{np}}\right].$$

- Poisson equations for δe^{-4A} and B_m are solved respectively in terms of biscalar and bivector Green's functions. Using a relation between derivatives of these Green's functions, we can satisfy another constraint that we get which relates div B_1 to δe^{-4A}.

- To work with the action, there is an organising principle. Schematically consider an action for some set fields ψ_a:

$$S = \int d^4x d^6y \; \mathscr{L}(\psi_a, \partial_M \psi_a, \partial_M \partial_N \psi_a) \,, \qquad \text{EOM}: \; E_a = 0 \quad a = 1,2,\ldots$$

- Consider a background: $\psi_a^{(0)} : E^a_{(0)} = 0$. Linear fluctuations around background values give:
$\psi_a = \psi_a^{(0)} + \delta\psi_a \implies E^a = E^a_{(0)} + \delta E^a$ where δE^a involve $\delta\psi_b$, $\partial_M \delta\psi_b$, $\partial_M \partial_N \delta\psi_b$.

- At zeroth order in the action we get a constant. At linear order: vanishes by background EOM.
At second order: $S = \frac{1}{2} \int d^4x d^6y \; \delta\psi_a \; \delta E^a$.

- δE^a are already computed [previous slides]. Plug them in above expression. Impose the constraints. Integrate y. [Subtlety regarding the C_4 fluctuations are dealt with keeping only magnetic components.]

- For the case of geometric Kähler moduli [including volume modulus and C_4 axions]:

$$4\kappa_{10}^2 S = 3\tilde{V}^{(0)} \int d^4x e^{2\Omega^{(0)}} \left[(C^{-1})^{IJ} u_I \Box u_J + C^{IJ} b_I \Box b_J\right]$$

$$3\tilde{V}^{(0)}\,(C^{-1})^{IJ} = \mathcal{G}^{IJ}, \qquad \mathcal{G}^{IJ} = \int_6 e^{-4A^{(0)}} \omega_2^I \wedge \tilde{\star}^{(0)} \omega_2^J + \frac{i}{2\mathrm{Im}\,\tau^{(0)}} \int_6 \left(\Lambda_1^I \wedge \bar{G}_3^{(0)} - \bar{\Lambda}_1^I \wedge G_3^{(0)}\right) \wedge \omega_2^J$$

$\tilde{V}^{(0)}$: volume of the background CY. Λ_1^I satisfies a Poisson equation sourced by $\tilde{\star}^{(0)}\,(G_3^{(0)} \wedge \omega_2^I)$.

- Recall : $\delta \tilde{J}_2 = u_I \omega_2^I, \qquad \delta C_4 \supset b_I\, \tilde{\star}^{(0)}\, \omega_2^I$

  ω_2^I forms a basis of harmonic $(1,1)$ forms such that $\int_6 \omega_2^I \wedge \tilde{\star}^{(0)} \omega_2^J = 3\tilde{V}^{(0)} \delta^{IJ}$.

  $u_1 =$ volume modulus. $\omega_2^1 = \tilde{J}_2^{(0)}$ which is the Kähler form of the background.

- C^{IJ} depends on background warp factor, 3-form flux, axio-dilaton and implicitly on the background metric $\tilde{g}_{mn}^{(0)}$. E.g. which representative of the cohomology is harmonic depends on the metric $\tilde{g}_{mn}^{(0)}$.

- In a similar way we have obtained the 4D effective theory of flat complex structure moduli.

- There seems to have kinetic mixing of the form $\mathscr{G}^{Ia}u_I\hat{\partial}^{\hat{2}}u_a$ between Kähler moduli u_I and flat complex structure moduli u_a. We could not prove $\mathscr{G}^{Ia}$ vanish. However these contributions drop in large volume limit.

IV. Outlook

Develop computational tools capable of calculating the moduli space metric as the background CY metric is varied over the moduli space.

This would allow for precision string phenomenology even in the case of strong warping.

Notably, there are recent advancements in numerically computing the metric on CYs and harmonic forms on them.

The dimensional reduction of the massive complex structure moduli remains open. For conifold deformation modulus first steps appear in [Lüst-Randall '22].

G_3 needs a shift violating linearised ISD condition. $\tilde{F}_5$ ansatz needs modification.

The tadpole conjecture [Bena et al '20 '21, Lüst '21, Plauschinn '21,…] suggests an upper bound on the number of stabilized complex structure moduli for compactifications on CY orientifolds with large Hodge number $h^{2,1}$. These tests had been conducted using the 4D superpotential [or period vector].

For a given CY orientifold and flux quanta satisfying the D3 tadpole bound, determining which complex structure deformations satisfy the linearised ISD condition poses a counting problem. We have demonstrated this in the simpler case of a toroidal orientifold $T^6/\mathbb{Z}_2$ [which is not under the scope of the conjecture] agreeing with [Cicoli-Licheri-RM-Maharana '22].

We have seen that our approach is equivalent to using the 4D superpotential but offers an alternative formulation that may provide insight into proving the conjecture or finding counterexamples [if any] by scanning over flux vacua.

What about Kaluza-klein decomposition in Warped compactifications? For first steps, see [Shiu-Torroba-Underwood-Douglas '08]

Compute tree-level four-graviton amplitudes in warped flux compactifications where dilaton is a modulus (we have obtained such backgrounds in *[A]*). This will advance our understanding of the classical Regge growth conjecture for tree-level amplitudes *[B]*. I.e., amplitudes do not grow faster than $\mathcal{O}(s^2)$. We believe progress can be made within 10d supergravity (+local sources) by generalizing similar works in the 5d Randall-Sundrum (RS) scenario *[C]*.

A. [Cicoli-Licheri-RM-Maharana '22], also in current work

B. The Future of String Theory: 100 Open Questions, Strings 2024. See problem posed by S. Minwalla.

C. [Chivukula-Foren-Mohan-Sengupta-Simmons '19], and their follow up papers.

Thank You For Your Attention.